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# CONTINUOUS SINGLE-VALUED STRATEGIES IN EVASION PROBLEMS†

S. A. BRYKALOV

Ekaterinburg

e-mail: brykalov@imm.uran.ru

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Continuous methods of control with negative feedback are considered in evasion problems when there is interference. The state of the controlled system is characterized by a finite-dimensional vector. The dynamics is described by an ordinary differential equation which is linear in the phase vector. The control parameter can occur in the equation in a non-linear manner and, also, in the coefficient of the phase vector. The differential equation also contains the unknown interference. It is assumed that the control consists of evasion from a convex, closed target set which is specified in the functional space of the trajectories of the system. In particular, this formation of the problem contains the case of a target set in a finite-dimensional state space of the system at the right-hand end of an interval. Control methods are studied which are described by single-valued mappings, which depend continuously on the phase vector. These control methods can use the deviation of the argument. In the case of natural constraints imposed on the system, it is shown that, if a certain continuous, negative feedback, control method guarantees evasion for any permissible interference, then a control method without negative feedback can be found which also guarantees the deviation. 

In the theory of positional differential games [1–4], the strategy of a game is a function which describes the negative feedback in the conflicting controlled system. The properties of strategies which are continuous in the phase vector have been discussed in detail ([1, §55], [2, §3], [3, pp. 232–239], and [5–8]). Continuous strategies in a differential game with an equation which is linear in the phase vector have been considered under the assumption that the coefficient of the phase vector is independent of the control.

A situation is studied below when this dependence can exist and, furthermore, these results are extended from the initial problems to certain boundary-value problems. At the same time, we have succeeded in giving a fairly simple proof of a theorem on continuous strategies, based on the application of Kakutani's theorem in the space of continuous functions and a lemma on the closure of the graph of a multivalued mapping. Here, simplifying assumptions (convexity of the target set, single-valuedness of the strategies and certain other assumptions) imposed below are used. Considerations on how to discard or relax these simplifying assumptions due to the complexity of the proof can be found in [7, 8] which are based on the methods of algebraic topology.

## 1. FORMULATION OF THE PROBLEM

We shall investigate a controlled system which is described by the differential equation

$$\dot{x} = A(t, \upsilon)x + g(t, u, \upsilon)$$

The independent variable $t \in [t_0, \vartheta]$ usually denotes the time, $x$ is the finite-dimensional phase vector, $\upsilon$ is the control and $u$ is the interference. The geometric constraints $u \in P$, $\upsilon \in Q$ are specified. A target set $M$ was fixed in the functional space of the trajectories of the system. It is required to act on the system using a control $\upsilon \in Q$ such that the evasion $x(\cdot) \notin M$ is guaranteed whatever the form of the interference $u(t) \in P$.

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The target set in the space of the trajectories naturally appears, in particular, as a set of the level of a non-terminal functional (which, for example, depends on the values of the solution at certain points or which contains a maximum, integral, and other non-local operations). Specific examples of differential games with non-terminal payment functionals have been considered in [4].

A situation is often encountered in which the target set $M_\vartheta$ is specified in a finite-dimensional space and the evasion game endeavours to satisfy the relation $x(\vartheta) \notin M_\vartheta$. This situation can be regarded as a special case of the problem with a target set $M$ in the space of the trajectories. It suffices to take as $M$ the set of all continuous functions terminating in $M_\vartheta$, that is, which satisfy the condition $x(\vartheta) \in M_\vartheta$.

It is possible to form the required negative-feedback control $\upsilon$ taking into account the measurement of the phase vector and, also, to use more general strategies with a memory $\upsilon = \upsilon(t, x(\cdot))$. It would be natural to attempt to treat strategies which possess the property of continuity with respect to the phase vector. However, in many problems, the possibilities of such strategies are restricted.

The following will be shown below for fairly general assumptions concerning the system. If it is possible to guarantee evasion by means of a strategy $\upsilon = \upsilon(t, x(\cdot))$ which is continuous in $x(\cdot)$, then evasion can be ensured using a preset control $\upsilon = \upsilon(t, y(\cdot))$, where $y(\cdot)$ is a suitable fixed function. Hence, if, in these problems, it is possible to evade using a continuous negative-feedback control, it is also possible to evade without negative feedback, that is, using a program.

The result is illustrated by the simple example of a differential game in which it is required that the origin of coordinates is avoided at a finite instant of time and the motion of the system is initially determined solely by the interference and subsequently only by the evading player. This leads to discontinuous coefficients in the equation.

Note that the mathematical result presented below is proved for the strategies $\upsilon = \upsilon(t, x(\cdot))$, containing both hysteresis as well as anticipation. This result also enables one to consider the case not only of the initial problem that also of certain boundary-value problems, which turns out to be useful if the independent variable has the meaning of a coordinate, rather than time. Strategies with variation of the argument and boundary conditions containing the control parameters arise in certain problems of the steady-state temperature distributions in a rod, the heating of which is controlled using continuous negative feedback (see [9]).

We shall use the following notation ($n$ is an integer, $n \geq 1$): $\mathbf{R}^n$ is the space of $n$-dimensional vectors (columns), the norm $|\cdot|_n$ of which is fixed, $\mathbf{R}^{n\times n}$ is the space of $n \times n$ matrices with real elements, the norm of which $|\cdot|_{n\times n}$ is matched with the vector norm being considered, that is, $|Ab|_n \leq |A|_{n\times n}|b|_n$ for arbitrary $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$, $C^0$ is the space of continuous functions, $L_1$ is the space of Lebesgue measurable functions with an integrable modulus (with an integral norm $|x(t)|_n$ or $|x(t)|_{n\times n}$ for the functions $x(t)$ with values in the space of $n$-dimensional vectors or $n \times n$ matrices) and $AC$ is the space of absolutely continuous functions.

The usual norms of the enumerated functional spaces are used, in particular,

$$\|x(\cdot)\|_{AC} = \|x(\cdot)\|_{C^0} + \|\dot{x}(\cdot)\|_{L_1}$$

The sign co corresponds to a convex shell and cl co to a convex closed shell.

## 2. SYSTEMS WITH CONTINUOUS NEGATIVE FEEDBACK

We fix certain real numbers $t_0 < \vartheta$ and integers $n, p, q \geq 1$. If nothing else follows from the context, the functional spaces which are used consist of functions defined in the interval $[t_0, \vartheta]$ and which take values in $\mathbf{R}^n$. For example, $C^0$ denotes $C^0([t_0, \vartheta], \mathbf{R}^n)$, unless otherwise stated. We assume that the sets $P \subset \mathbf{R}^p$, $Q \subset \mathbf{R}^q$ are non-empty and closed. Furthermore, the set $P$ is bounded. The function $g : [t_0, \vartheta] \times P \times Q \to \mathbf{R}^n$ satisfies the Carathéodory conditions. This means that, for almost every fixed $t$, the function $g(t, u, \upsilon)$ is continuous in $u$ and $\upsilon$ and measurable in $t$ for any fixed $u$ and $\upsilon$. Furthermore, suppose a function $\xi : [t_0, \vartheta] \to [0, \infty)$, $\xi(\cdot) \in L_1$ is found such that the inequality

$$|g(t, u, \upsilon)|_n \leq \xi(t) \tag{2.1}$$

is true for almost all $t \in [t_0, \vartheta]$ and all $u \in P$, $\upsilon \in Q$. The matrix function $A : [t_0, \vartheta] \times Q \to \mathbf{R}^{n\times n}$ satisfies the Carathéodory conditions. The limit

$$|A(t, \upsilon)|_{n\times n} \leq \eta(t) \tag{2.2}$$

is satisfied for a certain $\eta : [t_0, \vartheta] \to [0, \infty)$, $\eta(\cdot) \in L_1$ for almost every $t \in [t_0, \vartheta]$ and any $\upsilon \in Q$. The set $M \subset C^0$ is convex and closed. Suppose $\upsilon : [t_0, \vartheta] \times C^0 \to Q$ satisfies the Carathéodory conditions. Thus $\upsilon(t, z(\cdot))$ in the case of almost every fixed number $t \in [t_0, \vartheta]$ depends continuously on the function $z(\cdot) \in C^0$ and, for every fixed function $z(\cdot)$, the expression $\upsilon(t, z(\cdot))$ is measurable with respect to the variable $t$. The mappings $\sigma : C^0 \to [t_0, \vartheta]$ and $h : C^0 \to \mathbf{R}^n$ are continuous. A number $K \geq 0$ is found such that, for every function $z(\cdot) \in C^0$, the inequality

$$|h(z(\cdot))|_n \leq K \tag{2.3}$$

is true.

*Remark* 1. Actually, the boundedness of $h$ solely in the set $M$ is sufficient. However, the assumption that $h$ is bounded in the whole of the space $C^0$ enables one to simplify the proof of the theorem presented below to some extent.

We will now agree that $g(t, P, r)$ is the set of all vectors of the form $g(t, \alpha, r)$, where the number $t$ and the vector $r$ are fixed and $\alpha$ runs through the set $P$.

*Theorem.* Suppose the boundary-value problem

$$\dot{x}(t) \in A(t, v(t, x(\cdot)))x(t) + \operatorname{co} g(t, P, v(t, x(\cdot))) \tag{2.4}$$

$$x(\sigma(x(\cdot))) = h(x(\cdot)) \tag{2.5}$$

does not have solutions $x(\cdot) \in M \cap AC$. A function $y(\cdot) \in C^0$ is then found such that the initial problem

$$\dot{x}(t) \in A(t, v(t, y(\cdot)))x(t) + \operatorname{co} g(t, P, v(t, y(\cdot))) \tag{2.6}$$

$$x(\sigma(y(\cdot))) = h(y(\cdot)) \tag{2.7}$$

also does not have solutions $x(\cdot) \in M \cap AC$.

*Remark* 2. We stress that, in the theorem, the point is that there are no solutions belonging to the set $M \cap AC$. In the case of the requirements imposed on $A$, $v$, $g$, $P$, $\sigma$ and $h$, the problems being considered necessarily have solutions in $AC$. It is simplest to show this as follows. We specify a certain element $\beta \in P$ and note that, for any function $y(\cdot)$, the linear ordinary differential equation

$$\dot{x}(t) = A(t, v(t, y(\cdot)))x(t) + g(t, \beta, v(t, y(\cdot)))$$

has a solution, which obeys initial condition (2.7). This solution also satisfies the initial problem (2.6), (2.7). The solvability of boundary-value problem (2.4), (2.5) now follows from the theorem formulated above if the whole of the space $C^0$ is taken as $M$.

*Remark* 3. Since the bounded closed sets in finite-dimensional space become convex on the right-hand sides of the differential inclusions (2.4) and (2.6), the convex envelope is identical to a convex closed envelope. This follows from Carathéodory's theorem (see [10, pp. 155 and 158], for example).

*Remark* 4. The transition to a convex closed envelope on the right-hand sides of the inclusions (2.4) and (2.6) becomes unnecessary if the following requirement for the convexity of the vectogram is satisfied: the set $g(t, P, r)$ is convex for almost every $t \in [t_0, \vartheta]$ and every $r \in Q$.

The following is required to prove the theorem.

*Lemma on a closed graph.* Suppose we are given a multivalued mapping

$$[t_0, \vartheta] \times C^0 \times C^0 \ni (t, x(\cdot), z(\cdot)) \mapsto F(t, x(\cdot), z(\cdot)) \subset \mathbf{R}^n$$

where the set $F(t, x(\cdot), z(\cdot))$ is convex and closed for any $x(\cdot), z(\cdot) \in C^0$ and almost every $t \in [t_0, \vartheta]$, the multivalued mapping

$$C^0 \times C^0 \ni (x(\cdot), z(\cdot)) \mapsto F(t, x(\cdot), z(\cdot)) \subset \mathbf{R}^n$$

is semicontinuous from above for almost every fixed $t \in [t_0, \vartheta]$ and, for every $N \geq 0$, a function $\gamma_N$: $[t_0, \vartheta] \to [0, \infty)$, $\gamma_N(\cdot) \in L_1$ is found for every $N \geq 0$ such that the inequality $|y|_n \leq \gamma_N(t)$ holds for any $x(\cdot), z(\cdot)$ from the sphere $\|x(\cdot)\|_{C^0}$, $\|z(\cdot)\|_{C^0} \leq N$ and for almost every $t$ and every $y \in F(t, x(\cdot), z(\cdot))$.

For $z(\cdot) \in C^0$, we denote the set of all $x(\cdot) \in AC$, which satisfy the differential inclusion $\dot{x}(t) \in F(t, x(\cdot), z(\cdot))$ almost everywhere, by $\Omega(z(\cdot))$. Then,

$$C^0 \ni z(\cdot) \mapsto \Omega(z(\cdot)) \subset C^0 \tag{2.8}$$

is a closed graph.

*Remark* 5. The conditions of the lemma on a closed graph do not exclude the degenerate case. When this lemma is used, additional requirements are necessary which ensure the existence of the functions $z(\cdot)$ for which the sets $\Omega(z(\cdot))$ are non-empty.

*Remark* 6. It is well known that, in the case of a multivalued mapping with non-empty closed images, which acts from a metric space into a compact metric space, closure of the graph is equivalent to semicontinuity from above (see [11, p. 133], for example). In many cases, this enables one to reformulate the corresponding requirement on $F$ in the condition of the lemma.

Proof of the lemma. Suppose $x_i(\cdot) \to x_\infty(\cdot)$, $z_i(\cdot) \to z_\infty(\cdot)$ in the space $C^0$ when $i \to \infty$ and $x_i(\cdot) \in \Omega(z_i(\cdot))$ for all natural $i$. It is necessary to show that

$$x_\infty(\cdot) \in \Omega(z_\infty(\cdot)) \tag{2.9}$$

The convergent sequences $x_i(\cdot)$, $z_i(\cdot)$ are bounded by a certain quantity: $\|x_i(\cdot)\|_{C^0}$, $\|z_i(\cdot)\|_{C^0} \le N$. It follows from the requirements which have been imposed that $|\dot{x}_i(t)|_n \le \gamma_N(t)$ for all natural $i$ and almost every $t$. Hence

$$|x_i(t_2) - x_i(t_1)|_n \le \left| \int_{t_1}^{t_2} \gamma_N(\tau) d\tau \right|$$

for all $t_1, t_2 \in [t, \vartheta]$ and natural $i$. It is clear from the last inequality that it is also satisfied in the case of the limiting function $x_\infty(\cdot)$. Thus, the function $x_\infty(\cdot)$ is absolutely continuous ([12, pp. 141 and 226]) and, consequently, has a derivative from $L_1$.

It is well known [13, pp. 295 and 296] that the weak $L_1$ convergence $\dot{x}_i(\cdot) \to \dot{x}_\infty(\cdot)$ follows from the pointwise convergence

$$\int_{t_0}^{(\cdot)} \dot{x}_i(\tau) d\tau \to \int_{t_0}^{(\cdot)} \dot{x}_\infty(\tau) d\tau$$

and the boundedness of the norm $\|\dot{x}_i(\cdot)\|_{L_1} \le \|\gamma_N(\cdot)\|_{L_1}$. According to Mazur's theorem [14, p. 120], the weak limit of a sequence can be approximated to any accuracy by means of a certain convex combination of a finite set of terms of the sequence, selected according to the accuracy which is required. It is obvious that, neglecting a finite number of the first terms in a weakly converging sequence, we again obtain a weakly converging sequence. Hence, using $\dot{x}_i(\cdot)$, it is possible to construct a sequence $y_i(\cdot)$ such that $y_i(\cdot) \to \dot{x}_\infty(\cdot)$ in the space $L_1$, and, at the time,

$$y_i(t) = \sum_{k=i}^{n_i} p_{ki} \dot{x}_k(t) \tag{2.10}$$

are convex combinations, that is

$$p_{ki} \ge 0, \quad \sum_{k=i}^{n_i} p_{ki} = 1$$

The convergence accordingly follows from the convergence of the sequence $y_i(\cdot)$ in the space $L_1$. This means that the given sequence converges almost everywhere [12, p. 96]. In order to simplify the notation, we shall agree to assume that the sequence $y_i(\cdot)$ converges almost everywhere.

So, on discarding a denumerable family of sets of zero measure from the interval, we obtain the following. For almost all $t \in [t_0, \vartheta]$, equality (2.10) is satisfied for every natural $i$, the sequence $y_i(t)$ converges to $\dot{x}_\infty(t)$, the set $F(t, x_\infty(\cdot), z_\infty(\cdot))$ is convex and closed, the multivalued mapping

$$C^0 \times C^0 \ni (x(\cdot), z(\cdot)) \mapsto F(t, x(\cdot), z(\cdot)) \subset \mathbf{R}^n$$

is semicontinuous from the above and $\dot{x}_i(t) \in F(t, x_i(\cdot), z_i(\cdot))$ for all natural $i$.

We specify any of these $t$ and a certain $\varepsilon > 0$. Suppose $B_\varepsilon$ is a closed sphere in $\mathbf{R}^n$ of radius $\varepsilon$ with centre at zero. By virtue of the semicontinuity from above, a $\delta > 0$ is found such that, for all $x(\cdot)$, $z(\cdot)$ for which $\|x(\cdot) - x_\infty(\cdot)\|_{C^0} < \delta$, $\|z(\cdot) - z_\infty(\cdot)\|_{C^0} < \delta$, we have

$$F(t, x(\cdot), z(\cdot)) \subset F(t, x_\infty(\cdot), z_\infty(\cdot)) + B_\varepsilon$$

(Note that the set $F(t, x(\cdot), z(\cdot))$ may turn out to be empty for some of these $x(\cdot)$, $z(\cdot)$.) Hence, for a certain $i_0$, if $i \geq i_0$, we have

$$\dot{x}_i(t) \in F(t, x_i(\cdot), z_i(\cdot)) \subset F(t, x_\infty(\cdot), z_\infty(\cdot)) + B_\varepsilon$$

The last set is convex as the pointwise sum of two convex sets. Recalling that $y_i(t)$ is a convex combination of the vectors $\dot{x}_i(t), \ldots, \dot{x}_{n_i}(t)$, we obtain the following: if $i \geq i_0$, then the inclusion

$$y_i(t) \in F(t, x_\infty(\cdot), z_\infty(\cdot)) + B_\varepsilon$$

is satisfied.

The right-hand side of this relation is closed as the pointwise sum of a closed and a compact set. Also, since $y_i(t) \to \dot{x}_\infty(t)$, we obtain

$$\dot{x}_\infty(t) \in F(t, x_\infty(\cdot), z_\infty(\cdot)) + B_\varepsilon$$

Here, the first term is closed and the second term is a sphere of arbitrarily small radius. Hence,

$$\dot{x}_\infty(t) \in F(t, x_\infty(\cdot), z_\infty(\cdot))$$

Since, here, $t$ is almost any $t$ from $[t_0, \vartheta]$, we obtain inclusion (2.9).

*The theorem is proved by* reductio ad absurdum. Suppose the conclusion of the theorem is not true. Then, for any function, $y(\cdot) \in C^0$, the initial problem (2.6), (2.7) has just one solution $x(\cdot) \in M \cap AC$. We denote the set of all absolutely continuous solutions of the initial problem (2.6), (2.7) corresponding to a given continuous function $y(\cdot)$ by $\Psi(y(\cdot))$. Hence, the set $\Psi(y(\cdot)) \cap M$ is non-empty for every function $y(\cdot) \in C^0$. Furthermore, this set is convex by virtue of the linearity of the system with respect to the phase vector and the convexity of the set $M$.

We now make use of the lemma on a closed graph, putting

$$F(t, x(\cdot), z(\cdot)) = A(t, \upsilon(t, z(\cdot)))x(t) + \operatorname{co} g(t, P, \upsilon(t, z(\cdot)))$$

The requirement concerning the convexity and the closedness of the set $F(t, x(\cdot), z(\cdot))$ is satisfied. It follows from the Carathéodory conditions that, for almost every specified $t \in [t_0, \vartheta]$, the mappings $A(t, \upsilon)$, $g(t, u, \upsilon)$, $\upsilon(t, z(\cdot))$ are continuous in the remaining arguments. Using the well-known properties of multivalued mappings (see [11, pp. 137 and 138], for example), we conclude that, for these values of $t$, the multivalued mapping $(x(\cdot), z(\cdot)) \mapsto F(t, x(\cdot), z(\cdot))$ is semicontinuous from above. By virtue of inequalities (2.1) and (2.2), the vectors from the set $F(t, x(\cdot), z(\cdot)) \subset \mathbf{R}^n$ according to the norm do not exceed $\eta(t)\ \|x(\cdot)\|_{C^0} + \xi(t)$ for almost all $t$ and all $x(\cdot)$, $z(\cdot)$ and it is possible to put $\gamma_N(t) = \eta(t)N + \xi(t)$.

Hence, all the conditions of the lemma are satisfied and (2.8) therefore has a closed graph, where $\Omega(y(\cdot))$ is the set of all absolutely continuous solutions of the differential inclusion (2.6). On taking account of the continuity of the mappings of $\sigma$ and $h$ in the initial conditions (2.7), we see that

$$C^0 \ni z(\cdot) \mapsto \Psi(z(\cdot)) \subset C^0$$

also has a closed graph. Finally, by making use of the fact that the set $M \subset C^0$ is closed, we come to the conclusion that the graph of the multivalued mapping

$$C^0 \ni z(\cdot) \mapsto \Psi(z(\cdot)) \cap M \subset C^0$$

is closed.

We will show that the set $\cup_{z(\cdot)}\Psi(z(\cdot))$, where the union is taken over all functions $z(\cdot) \in C^0$, has a compact closure in the space $C^0$.

Actually, if $x(\cdot) \in \Psi(y(\cdot))$ for a certain function $y(\cdot)$, then, by virtue of relations (2.6), (2.1) and (2.2), we have

$$|\dot{x}(t)|_n \leq \eta(t)|x(t)|_n + \xi(t) \leq \chi(t)(1 + |x(t)|_n); \quad \chi(t) = \eta(t) + \xi(t), \quad \chi(\cdot) \in L_1$$

for almost all $t \in [t_0, \vartheta]$.

On taking account of relations (2.7) and (2.3), we obtain

$$|x(t)|_n \le K + \left| \int_{\sigma(y(\cdot))}^{t} \chi(\tau)(1 + |x(\tau)|_n) d\tau \right|$$

for all $t \in [t_0, \vartheta]$. It follows from this (see [15, pp. 189 and 190], for example) that a number $H \ge 0$ exists such that the inequality $|x(t)|_n \le H$ is satisfied for any $y(\cdot) \in C^0$, $x(\cdot) \in \Psi(y(\cdot))$, $t \in [t_0, \vartheta]$. Then, $|\dot{x}(t)|_n \le \chi(t)(1 + H)$ almost everywhere and, for any $t_1$, $t_2$, we obtain

$$|x(t_2) - x(t_1)|_n \le (1 + H) \left| \int_{t_1}^{t_2} \chi(\tau) d\tau \right|$$

The compactness of the closure of the set $\cup_{z(\cdot)} \Psi(z(\cdot)) \subset C^0$ follows from the Arzelà–Ascoli theorem [13, p. 48] and the properties of the Lebesgue integral [12, p. 141].

The closed convex envelope $S = \mathrm{cl\ co}(M \cap \cup_{z(\cdot)} \Psi(z(\cdot)))$, where $z(\cdot)$ runs through the whole space $C^0$, is a non-empty, convex compactum (see [13, p. 105] and [15, p. 164]). The multivalued mapping $z(\cdot) \mapsto \Psi(z(\cdot)) \cap M$ translated $S$ into itself and has non-empty, convex values. At the same time, the graph of this mapping is closed in the sense of the norm of the Banach space $C^0$. By Kakutani's theorem [13, p. 360], just a single fixed point $x(\cdot) \in \Psi(x(\cdot)) \cap M$ exists. This function $x(\cdot)$ is an absolutely continuous solution of boundary-value problem (2.4), (2.5) lying in the set $M$, which is impossible according to the condition of the theorem. The contradiction obtained completes the proof of the theorem.

*Remark* 7. If the matrix $A(t, \upsilon) = A(t)$, $A(\cdot) \in L_1$ is independent of $\upsilon$, the theorem obtained can be extended to the case of multivalued strategies. This has been done previously [8, Corollary 2] for initial conditions of the form $x(t_0) = x_0$.

*Remark* 8. Classical motions, that is, absolutely continuous functions which satisfy the differential inclusion being considered almost everywhere, are used in the theorem which has been formulated above. In the case of the initial conditions $x(t_0) = x_0$ of the positional strategies $\upsilon = \upsilon(t, x(t))$, it is also possible to apply this theorem to structural motions [3, pp. 11–19] which are the uniform limits of stepwise motions in refining nets. The constructions here will be analogous to the corresponding construction [8, Section 2], where the matrix $A(t, \upsilon) = A(t)$ was independent to $\upsilon$ (see [8, Corollary 4]).

*Remark* 9. Under the assumption that relations (2.5) and (2.7) take the form of the initial conditions $x(t_0) = x_0$, the theorem presented above can be obtained using results obtained earlier [8]. This method of proving a special case of the theorem requires the use of certain concepts in algebraic topology and is based, in the final analysis, on the Eilenberg–Montgomery theorem on a fixed point which is associated with groups of homologies. A detailed discussion of the corresponding properties of functional-differential inclusions can be found in [7].

## 3. AN EXAMPLE OF A CONFLICT CONTROL SYSTEM

We will now illustrate the theorem using a simple differential game.

Consider the differential equation

$$\dot{x} = a(t)u + (1 - a(t))\upsilon, \quad 0 \le t \le 2; \quad a(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t \le 2 \end{cases} \tag{3.1}$$

where the vectors $x$, $u$ and $\upsilon$ have the dimension $n \ge 1$.

When $0 \le t < 1$, Eq. (3.1) therefore has the form $\dot{x} = u$ and the motion is determined by the interference $u$. When $1 \le t \le 2$, Eq. (3.1) takes the form $\dot{x} = \upsilon$ and the evading player controls the motion. The geometrical constraints $|u|_n \le 1$, $|\upsilon|_n \le 1$ must be satisfied. We consider realizations of the controls by measurable functions of the argument $t$ and that the solution is absolutely continuous and satisfies Eq. (3.1) almost everywhere A null initial condition

$$x(0) = 0 \tag{3.2}$$

is specified. In the presence of unknown interference $u = u(t)$, it is required, by choosing the control $\upsilon$, to maximize $|x(2)|_n$, where $x$ is the solution of the initial problem (3.1), (3.2).

If the function $x$ is the solution of problem (3.1), (3.2), then

$$x(2) = \int_0^1 u d\tau + \int_1^2 \upsilon d\tau$$

In this differential game, it is impossible to guarantee evasion of the origin of the coordinates using a preset control $\upsilon = \upsilon(t)$, where $\upsilon(\cdot) \in L_1$ and $|\upsilon(t)|_n \le 1$ almost everywhere. In fact, it is sufficient to put

$$u(t) \equiv u = -\int_1^2 \upsilon(\tau) d\tau$$

Then, $|u|_n \le 1$ and $x(2) = 0$.

On the other hand, there is a simple method of evasion which only requires a single measurement of the phase vector and is described by a continuous mapping. When $1 \le t \le 2$, we put $\upsilon(t) \equiv |x(1)|_n^{-1} x(1)$ if $x(1) \ne 0$ and we take $\upsilon(t) \equiv \upsilon_0$ if $x(1) = 0$. Here $\upsilon_0$ is a certain fixed vector with the property $|\upsilon_0|_n = 1$. (It is obvious that, when $0 \le t < 1$, the choice of $\upsilon(t)$ in the unit sphere $|\upsilon|_n \le 1$ can be arbitrary.) This method of forming $\upsilon$ ensures that the inequality $|x(2)|_n \ge 1$ is satisfied. It is impossible to guarantee a value of $|x(2)|_n$ greater than unity for any control procedure whatsoever since it can turn out that $x(1) = 0$ and the control $\upsilon$ must satisfy the constraint $|\upsilon|_n \le 1$.

*Remark* 10. According to known results [1, 3], evasion in the differential game being considered can also be realized by means of a positional strategy and schemes with a step size which tends to zero. Such constructions have been described in detail earlier for another example of a differential game [2, pp. 18–21].

We will now show that, in the problem being considered, it is impossible to ensure evasion of the origin of the coordinates using control procedures which are continuous with respect to the phase vector. We now check that the theorem from Section 2 can be used. We put

$$t_0 = 0, \quad \vartheta = 2, \quad p = q = n$$

As $P = Q$, we take a closed sphere of unit radius with its centre at zero in the space $\mathbf{R}^n$. Suppose that

$$g(t, u, \upsilon) = a(t)u + (1 - a(t))\upsilon, \quad \xi \equiv 1, \quad A \equiv 0, \quad \eta \equiv 0, \quad \sigma \equiv 0, \quad h \equiv 0, \quad K = 0$$

Inequalities (2.1)–(2.3) are satisfied. The set of all points of discontinuity $g(t, u, \upsilon)$ is described by the relations $t = 1, u \ne \upsilon$. The function $g$ satisfies the Carathéodory conditions. We assume that the set $M$ consists of all continuous $z : [0, 2] \to \mathbf{R}^n$ such that $z(2) = 0$. Note that $M$ is convex and closed in the space $C^0$. We shall assume that the mapping $\upsilon(t, z(\cdot))$, where $t$ is a number and $z(\cdot)$ is a continuous function, satisfies the Carathéodory conditions and takes values on the sphere $Q$.

The differential inclusion (2.6) takes the form

$$\dot{x}(t) \in a(t)P + (1 - a(t))\upsilon(t, y(\cdot)) \tag{3.3}$$

It was verified above that, in the problem being considered, it is impossible to guarantee that zero will be evaded using preset controls. The initial problem (3.3), (3.2) therefore has a solution $x(\cdot) \in M \cap AC$ for any permissible choice of the mapping $\upsilon$ and the function $y(\cdot)$. The conclusion of the theorem from Section2 is not satisfied and the condition of the theorem, that is, that the functional-differential inclusion

$$\dot{x}(t) \in a(t)P + (1 - a(t))\upsilon(t, x(\cdot))$$

together with the null initial condition (3.2), should have a solution $x(\cdot) \in M \cap AC$ for any Carathéodory strategy $\upsilon$, is therefore also not satisfied.

Thus, it has been established that, in the problem being considered, the strategies $\upsilon$: $[0, 2] \times C^0 \to Q$, which satisfy the Carathéodory conditions, cannot guarantee the evasion of the origin of the coordinates. These strategies $\upsilon = \upsilon(t, x(\cdot))$ depend on $x(\cdot)$ as on a function specified in the whole of the interval [0, 2]. Similar control laws are encountered, for example, in problems associated with the control of the heating of a rod using the principle of negative feedback [9], where the independent variable has the meaning of a coordinate of a point in the rod.

The class of Carathéodory control laws $\upsilon = \upsilon(t, x(\cdot))$, $\upsilon : [0, 2] \times C^0 \to Q$ contains, in particular, strategies with memory, which are distinguished by the requirement of unpredictability (or physical feasibility). It is necessary to impose this if the independent variable $t$ is the time. This requirement excludes from consideration control laws which make use of information on the development of the process in the future.

We will now formulate the requirement of unpredictability. If the functions $x(\cdot), z(\cdot) \in C^0$ and the number $t \in [t_0, \vartheta]$ are such that $x(\tau) = z(\tau)$ for all $\tau \in [t_0, t]$, then the equality $\upsilon(\tau, x(\cdot)) = \upsilon(\tau, z(\cdot))$ is satisfied for almost all $\tau \in [t_0, t]$.

It should also be noted that the class being considered contains, in particular, the positional strategies $\upsilon = \upsilon(t, x(t))$, where the function $\upsilon : [0, 2] \times \mathbf{R}^n \to Q$ satisfies the Carathéodory conditions. Hence, in the problem being investigated, Carathéodory positional strategies also do not enable one to guarantee evasion.

*Remark* 11. If the discussion is restricted to strategies without an aftereffect, it is possible to consider the given game in the shorter interval $1 \le t \le 2$, assuming that the action of the interference lies in the choice of an initial position $x(1)$ which satisfies the condition $|x(1)|_n \le 1$. It is interesting to compare this with the fact that it is precisely when $t = 1$ when the measurement of the phase vector is carried out in the evasion procedure described above.

*Remark* 12. The existence of a deviation of the argument in the strategies and the discontinuity of the coefficients in Eq. (3.1) are among the special features of the simple example which has been analysed. These make it difficult to investigate using standard methods.

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*Translated by* E.L.S.